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FROM IRREDUNDANCE TO ANNIHILATION: A BRIEF OVERVIEW OF SOME DOMINATION PARAMETERS OF GRAPHS

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RESUMEN

Durante los últimos treinta años, el concepto de dominación en grafos ha levantado un interés impresionante. Una bibliografía reciente sobre el tópico contiene más de 1200 referencias y el número de definiciones nuevas está creciendo continuamente. En vez de intentar dar un catálogo de todas ellas, examinamos las nociones más clásicas e importantes (tales como dominación independiente, dominación irredundante, *k*-cubrimientos, conjuntos *k*-dominantes, conjuntos Vecindad Perfecta, ...) y algunos de los resultados más significativos.

PALABRAS CLAVES: Teoría de grafos, Dominación.

ABSTRACT

During the last thirty years, the concept of domination in graphs has generated an impressive interest. A recent bibliography on the subject contains more than 1200 references and the number of new definitions is continually increasing. Rather than trying to give a catalogue of all of them, we survey the most classical and important notions (as independent domination, irredundant domination, k-coverings, k-dominating sets, Perfect Neighborhood sets, ...) and some of the most significant results.

KEY WORDS: Graph theory, Domination.

INTRODUCTION

The idea of domination in a graph has existed far before the word of domination was precisely defined. For instance, the combinatorial problems, famous in the last century, of finding on a p x p chessboard the maximum number of pairwise non-attacking queens, the minimum number of queens sufficient to cover the chessboard and the minimum number of pairwise non-attacking queens sufficient to cover the chessboard are typically problems of independence, domination and independent domination. In 1958, Berge introduced the concept of dominating set under the term of external stable set (Berge 1962). In 1962, Ore used for the first time the word domination (Ore 1962). Since this time, a large amount of work has been done on the subject and a lot of related concepts have been introduced and studied (see the recent and very complete book Haynes *et al.* (1998 a) and its bibliography containing more than 1220 titles).

We recall here the definition and a few properties of the most usual domination related parameters, namely the parameters of domination, independence and irredundance, and introduce the more recent concepts of perfect neighborhood sets, *R*-annihilated irredundant sets and *R*-annihilated sets.

* Conferencia por invitación, dictada en las VIII Jornadas de Teoría de Grafos, Cumaná 1998.

First we specify our terminology. We consider only undirected simple graphs $G = (V, E)$ (in directed graphs the problems of domination are quite different since in particular, an independent dominating set, called *kernel,* does not always exist). We denote by n the order $|V|$ of G, by $N(x)$ (resp. $N[x] = N(x) \cup \{x\}$) the neighborhood (resp. closed neighborhood) of a vertex x, by $d(x) = |N(x)|$ the degree of x and by δ (resp. Δ) the minimum (resp. maximum) degree of G. For $X \subseteq V$, $N(X) = \cup_{x\in X} N(x)$, $N[X] = \cup_{x\in X} N[x] = X \cup N(X)$ and if $y \in V$, $N_X(y) = N(y) \cap X$. The subgraph $G[X]$ induced in G by a subset $X \subseteq V$ is often simply denoted by X.

The *X-private neighborhood* of a vertex $x \in X$ is the set $pn_X(x) = N[x] - N[X - \{r\}]$ and its elements are called the *X-private neighbors* of x. The X-private neighbors of x are thus x itself if x is isolated in X, and the *external private neighbors* which are the vertices of $V - X$ adjacent to x but to no other vertex of X. The vertex x is *irredundant* in X if $pn_X(x) \neq \phi$, *redundant* otherwise

To each subset $X \subseteq V$ we associate the partition $X \cup Y_X \cup Z_X \cup B_X \cup C_X \cup R_X$ of V where Z_X (resp. Y_X) is the set of isolated (resp. non isolated) vertices of X, B_X is the set of vertices of $V - X$ with exactly one neighbor in X (that is the set of the external X-private neighbors of the vertices of X), C_X is the set of the vertices of V with at least two neighbors in X, and $R_x = V - N[X]$ is the set of the vertices of $V - X$ with no neighbors in X. With this notation, X is *independent* if and only if $Y_X = \phi$.

DOMINATION, INDEPENDENCE AND IRREDUNDANCE

A subset X of V is a *dominating set* of G if every vertex of $V - X$ has at least one neighbor in X, in other terms, if $N[X] = V$ or equivalently, if $R_X = \phi$. More generally, if X and Y are subsets of V, we say that X dominates Y if $Y \subseteq N[X]$. Dominating sets always exist since V itself dominates G and the interesting notion is this one of minimal dominating sets which are dominating sets X such that $X - \{x\}$ is not dominating for all x in X. This fundamental concept of domination has been extended in different directions.

One can ask the dominating set to fulfil another condition and consider for instance connected dominating sets (X is connected), total dominating sets $(Z_X = \phi)$, independent dominating sets dominating cliques, dominating cycles, a.s.o. Note that in a connected graph G, independent, total or connected dominating sets always exist but that this is not true for dominating cliques or cycles. Hence the main subject of study of the second kind of dominating sets is their existence, while it is their size for the first kind.

$(Z_X = \phi)$, One can also consider different ways for a set X to dominate G. For instance, X is a k-covering (Meir and Moon 1975) if every vertex of $V - X$ is at distance at most k from some vertex of X, and a k-dominating set (Fink and Jacobson 1985) if each vertex of $V - X$ has at least k neighbors in X.

Let us come back to the notion of independent dominating set. It is easy to see that an independent set X is maximal under inclusion (that is X È $\{x\}$ is not independent for all x in V – X) if and only if X dominates G, and that in this case, X is a minimal dominating set of G. The minimum and maximum cardinalities of a maximal independent set of G are respectively denoted by $i(G)$ and $\beta(G)$. The minimum and maximum cardinalities of a minimal dominating set of G are respectively denoted by $\gamma(G)$ and $\Gamma(G)$ (we can omit G when there is no ambiguity). From the above characterization of maximal independent sets, every graph G satisfies $\gamma(G) \le i(G) \le \beta(G) \le \Gamma(G)$.

For instance, let $V(G) = \{x, y, x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ and $E(G) = \{xy\} \cup \{xx_i, yy_i, yz_i, y_i z_i\}_{1 \le i \le 3} \cup \{y_1y_2, y_2y_3, y_3y_1, z_1z_2, z_2z_3, z_3z_1\}$. Then $\{x,y\}$ and $\{x_1,x_2,x_3, y_1, y_2y_3\}$ are respectively minimum and maximum minimal dominating sets, $\{x, y_1, z_2\}$ and $\{x_1, x_2, x_3, y_1, z_2\}$ are respectively minimum and maximum maximal independent sets, and thus $\gamma(G) = 2$, $i(G) = 3$, $\beta(G) = 5$, $\Gamma(G) = 6$.

Can we find a characterization of minimal dominating sets similar to the characterization of maximal independent sets? In the previous example, the dominating set $X = \{x_1,x_2,x_3,y_1,y_2,y_3\}$ is minimal because each x_i being isolated in X, $X - \{x_i\}$ does not dominate x_i, and each y_i admitting an X-external private neighbor z_i, $X - \{y_i\}$ does not dominate z_i. The reason of the minimality of X is thus that each vertex of X is irredundant in X. A subset X of V is said to be *irredundant* if each vertex of X is irredundant in X. It is now easy to check that a dominating set X of G is minimal if and only if it is irredundant. Moreover, if X is both dominating and irredundant, then for all y in $V - X$, $N[y]$ Í $N[X]$ and thus X È $\{y\}$ is no more irredundant. This is the definition of a maximal irredundant set. The minimum and maximum cardinalities of a maximal irredundant set are respectively denoted by $ir(G)$ and $IR(G)$. Therefore, we get the following implications (1) and consequently the inequality chain (2) which was first given in Cockayne *et al.* (1978):

maximal independent	$\Leftrightarrow$	independent and dominating	$\Rightarrow$	minimal dominating	$\Leftrightarrow$	dominating and irredundant	$\Rightarrow$	maximal irredundant	(1)

$$1 \le ir(G) \le \gamma(G) \le i(G) \le \beta(G) \le \Gamma(G) \le IR(G) \qquad (2)$$

Since each subset of V containing exactly one vertex is irredundant, we are lead to consider for irredundant sets the same problem as before for independent and for dominating sets: how characterize the property for an irredundant set to be maximal? Given a set X Í V, we say that a vertex y of $V - X$ *annihilates* a vertex x of X if $\phi \ne \mathrm{pn}_X(x)$ Í $N[y]$. This means that x has X-private neighbors but no (X È $\{y\}$)-private neighbors. Therefore x is irredundant in X but redundant in X È $\{y\}$. The set X is *annihilated* by Y Í V if each vertex y of Y annihilates some vertex of X. Consider for instance the graph H given by $V(H) = \{x,y,x',y',x'',y'',z,t\}$ and $E(H) = \{xy, xx', x'x'', yy', y'y'', zx, zy, zt, tx'\}$. The set $X = \{x,y\}$ is irredundant since $\mathrm{pn}_X(x) = \{x'\} \ne \phi$ and $\mathrm{pn}_x(y) = \{y'\} \ne \phi$.. The set $X' = X \cup \{z\}$ is also irredundant since $\mathrm{pn}_{X'}(x) = \{x'\}$, $\mathrm{pn}_{x'}(y) = \{y'\}$ and $\mathrm{pn}_{X'}(z) = \{t\}$, and thus the irredundant set X is not maximal. However, the irredundant set X' is maximal. With the notation given in the introduction, $R_X = \{x'', y'', t\}$ and $N[\mathrm{R}_X] = \{x'' y'', t, x', y', z\}$, $R_{X'} = \{x'', y''\}$ and $N[R_{X'}] = \{x'',y'',x',y'\}$. The irredundant set X is not maximal because, the vertex z of $N[R_X]$ annihilating no vertex of X, the set X È $\{z\}$ is still irredundant. But every vertex of $N[R_{X'}]$ annihilates some vertex of X' (x' and x'' annihilate x, y' and y'' annihilate y). This is a general result and it is proved in Cockayne *et al.* (1997) that the irredundant set X is maximal if and only if it is $N[R]$-annihilated.

Before generalizing in the following two sections the $N[R]$-annihilated irredundant sets, let us point out that a lot of work has been done in relation to the six parameters involved in (2). As the determination of each of them is NP-complete, even in many restricted classes of graphs, any piece of information about them is interesting. When $ir(G) = 1$, that is when G has at least one *universal* vertex dominating G, then $\gamma(G) = i(G) = 1$. When $ir(G) \geq 2$, the only other inequality holding for all G between the six parameters is $\gamma(G) \leq 2\, ir(G) - 1$ (Allan and Laskar 1978 a; Bollobás and Cockayne 1979). Given any six integers m_i such that $2 \leq m_1 \leq m_2 \leq m_3 \leq m_4 \leq m_5 \leq m_6$ and $m_2 \leq 2m_1 - 1$, there exist graphs satisfying $ir(G) = m_1$, $\gamma(G) = m_2$, $i(G) = m_3$, $\beta(G) = m_4$, $\Gamma(G) = m_5$, $IR(G) = m_6$ (Cockayne and Mynhardt 1992). But if we restrict ourselves to particular classes of graphs, often defined by forbidden induced subgraphs, we can have stronger relations between the parameters. For instance, $\gamma(G) = i(G) \leq 3ir(G)/2$ in claw-free graphs, $\gamma(G) \leq 3ir(G)/2$ in block graphs or $\beta(G) = \Gamma(G) = IR(G)$ in bipartite or chordal graphs (see e.g. Allan and Laskar 1978 a; Cockayne *et al.* 1981; Faudree *et al.* 1998; Favaron *et al.* 1998; Jacobson and Peters 1990; Puech 1998 a; Puech 1998 b; Volkmann 1998; Zverovich 1998 ...). Also, many relations are known between these parameters and the order and the minimum or maximum degree of the graph. For instance, for all G, $ir(G) \geq 2n/3\Delta$, $n/(\Delta + 1) \leq \gamma(G) \leq n/2$ (if G has no isolated vertex), $i(G) \leq n - \Delta$, $IR(G) \leq n - \delta$, $i(G) \leq n + 2\delta - 2\sqrt{n\delta}$, $i(G) + IR(G) \leq 2n + 2\delta - \sqrt{2n\delta}$ (see e.g. Cockayne and Mynhardt 1997; Favaron 1988; Ore 1962; Sun and Wang 1999 ...). Another direction which raised up many results is the study of the behavior of these parameters under the deletion of a vertex or an edge, or the addition of an edge. On all these subjects, and other ones, the reader is refered to the bibliography of Haynes et al. (1998 a).

To finish this section, let us give as an example a table showing what is presently known on the value of the six previous parameters in chessboard graphs. The problems mentioned in the introduction and related to the positions of queens on a chessboard have been formalized in terms of domination, generalized to other chessboards and other domination parameters, and are vastely studied. The vertices of a chessboard graph are the p^2 squares of a p x p chessboard. For a given piece, the neighbors of a vertex x are the squares that such a piece placed in x can reach in one move. There are thus six chessboard graphs corresponding to the six chessmen (with an exception for the pawn which is replaced, because of its irregular moves, by a "pseudo-pawn" whose moves raise a grid Pp Pp). For each square of the table, we indicate what is known, namely the exact value of the corresponding parameter, or its order of magnitude when $p \to \infty$, or bounds. For the most recent results, a reference is given. Note that when the general value of a parameter is not known, there often exist results for small values of p (see Haynes *et al.* 1998 b, Chapter 6). And that some results also exist on $ir(N_p)$, $\gamma(N_p)$ and $i(N_p)$, but they are not yet definitive.

	ir	γ	i	β	Γ	IR
$Q\,p$ Queen		$\frac{p-1}{2} \leq \gamma \leq \frac{31p}{54} + O(1)$ (Burger *et al.* 1997)	$i \leq$ (Eisenstein *et al.* 1992)	$\frac{p^2}{5} + O(p)$	$\frac{5p}{2} + O(1) \leq \Gamma$ (Burger *et al.* 1997)	$IR \leq$ Burger *et al.* 1997)
$K\,p$ King	(Favaron *et al.* 1998)	$\left\lfloor \frac{p+2}{3} \right\rfloor^2$	$\left\lfloor \frac{p+2}{3} \right\rfloor^2$	$\left\lfloor \frac{p+2}{2} \right\rfloor^2$	$\frac{p^2}{3} + O(p)$ (Favaron *et al.* 1998)	$\frac{p^2}{3} + O(p)$ (Favaron *et al.* 1998)
$B\,p$ Bishop	p	p		$2p-2$	$2p-2$	$4p - 14$
$R\,p$ Rook						$2p - 4$
$N\,p$ Knight					$\left\lceil \frac{p^2}{2} \right\rceil$	$\left\lceil \frac{p^2}{2} \right\rceil$
$G\,p$ Grid	$\frac{p^2}{5} + O(p)$ (Favaron and Puech 1998)	$\frac{p^2}{5} + O(p)$ (Cockayne *et al.* 1985)	$\frac{p^2}{5} + O(p)$ (Favaron and Puech 1998)	$\left\lceil \frac{p^2}{2} \right\rceil$	$\left\lceil \frac{p^2}{2} \right\rceil$	$\left\lceil \frac{p^2}{2} \right\rceil$

PERFECT NEIGHBORHOOD SETS AND *R*–ANNIHILATED IRREDUNDANT SETS

Recently, perfect neighborhood sets were defined in Fricke *et al.* (1999). Given $X \subseteq V$, a vertex $v \in V$ is said to be *X-perfect* if $|N[v] \cap X| = 1$, that is if $v \in Z_X \cup B_X$. The set X is a *perfect neighborhood set* if each vertex of V is X-perfect or has an X-perfect neighbor, in other terms, if $Z_X \cup B_X$ is a dominating set of G. For instance, every maximal independent set is a perfect neighborhood set. The minimum and maximum cardinalities of a perfect neighborhood set are respectively denoted by $\theta(G)$ and $\Theta(G)$. Similarly, the minimum cardinality of an independent perfect neighborhood set is denoted by $\theta_i(G)$ and clearly, $\theta_i(G) \geq \theta(G)$. Fricke *et al.* (1999) proved that the two new parameters satisfy $\theta(G) \leq \gamma(G)$ and $\Theta(G) = \Gamma(G)$, and they conjectured that $\theta(G) \leq ir(G)$ for all G. In Favaron and Puech (1999), a family of counter-examples to this conjecture is given, showing that on the contrary, the difference $\theta(G) - ir(G)$ can be arbitrarily large (the smallest graph of this family has nearly 2 millions vertices). However, the inequality $\theta(G) \leq ir(G)$ holds in several families of graphs, among them trees and claw-free graphs (Cockayne *et al.* 1998; Favaron and Puech 1999).

A natural way to prove $\theta(G) \leq ir(G)$ for some graph G is to start from a maximal irredundant set X with $ir(G)$ vertices and to construct from X a perfect neighborhood set with at most $|X| = ir(G)$ vertices. Applying this method to claw-free graphs, we observed in Favaron and Puech (1999) that we could get a stronger result. First because it was possible to construct an independent perfect neighborhood set of the good size, so that $\theta_i(G) \leq ir(G)$. Secondly because we did not entirely use the property of the irredundant set X to be maximal, that is $N[R]$-annihilated, but only its weaker property to be R-annihilated. This observation lead us to introduce (Favaron and Puech 1999; Cockayne *et al.* 1998 a) the concept of *R-annihilated irredundant sets* (which we first called s*emi-maximal irredundant sets)* and to denote by $rai(G)$ the minimum cardinality of an R-annihilated irredundant set. Since every $N[R]$-annihilated set is R-annihilated, we clearly have $rai(G) \leq ir(G)$ for all G. Hence our proof in Favaron and Puech (1999) shows that $\theta(G) \leq \theta_i(G) \leq rai(G) \leq ir(G)$ if G is claw-free.

The example of the graph H given in Section 2 can help the reader to see the difference between $N[R]$–annihilated irredundant sets and R-annihilated irredundant sets. The irredundant set $X = \{x, y\}$ is R–annihilated since $R_X = \{x'', y'', t\}$ and x'' and t annihilate x, y'' annihilates y. But X is not $N[R]$–annihilated, and thus not maximal, since the vertex z of $N[R_x]$ annihilates nothing in X. However, we saw that the irredundant set $X' = \{x, y, z\}$ is $N[R]$–annihilated since maximal. Since H has no universal vertex, $rai(H) > 1$ and we can check that X' is a minimum maximal irredundant set. Hence, $rai(H) = 2$ and $ir(H) = 3$.

Perfect neighborhood sets can also be related to the 2-coverings and the 2-packings of G. Every perfect neigbborhood set X is a 2-covering (see Section 2) since every vertex of G is at distance at most 1 from $B_X \cup Z_X$ and B_X is dominated by X. Hence, if we denote by $\gamma_2(G)$ the minimum cardinality of a 2-covering, then $\gamma_2(G)$ q (G). In the other direction, a *k-packing,* introduced in Meir and Moon (1975), is a set of vertices such that any two of them are at distance more than k in G (in particular a 1-packing is an independent set). Every maximal 2-packing is an independent 2-covering satisfying $N(X) = B_X$, and thus is an independent perfect neighborhood set. Therefore, if we denote by $\rho_L(G)$ and $\rho(G)$ the minimum and maximum cardinalities of a maximal 2-packing, we have (G) $\rho_L(G)$ and thus (G) $\theta(G)$ (G) (G) (G) for all G. Moreover, it is proved in Meir and Moon (1975) that (G) (G).

N[*R*] –ANNIHILATED SETS AND *R*–ANNIHILATED SETS

After having reduced the class of the irredundant sets by considering the subclasses of the $N[R]$-annihilated irredundant sets and of the R-annihilated irredundant sets, we now enlarge the new classes by considering the $N[R]$-annihilated sets and the R-annihilated sets which are not necessarily irredundant.

We saw in Section 2 that an irredundant set is maximal if and only if it is $N[R]$-annihilated, and it is shown in Cockayne *et al* (1997) that in this case, it is a minimal (under inclusion) $N[R]$-annihilated set. Because the $N[R]$–annihilated sets were first called *external redundant* (Cockayne *et al* 1997; Cockayne *et al.* 1997), the minimum and maximum cardinalities of a minimal $N[R]$-annihilated set are respectively denoted by $er(G)$ and $ER(G)$. Hence, the sequences of implications (1) and of inequalities (2) can be enlarged as follows:

Maximal irredundant	$\Leftrightarrow$	irredundant and $N[R]$–annihilated	$\Rightarrow$	minimal $N[R]$–annihilated

and

$$\le er(G) \le ir(G) \le \gamma(G) \le i(G) \le \beta(G) \le \Gamma(G) \le IR(G) \le ER(G).$$

R-annihilated sets, defined in Cockayne *et al* (1998 a), both extend $N[R]$-annihilated sets and R-annihilated irredundant sets. Therefore, their minimum cardinality $ra(G)$ satisfies $ra(G) \leq er(G)$ and $ra(G) \leq rai(G)$ for all G. On the other hand, and as for the perfect neighborhood sets, every R-annihilated set X is a 2-covering since every vertex of B_x C_X is at distance 1 from X, and if R_X , every vertex v of Rx is at distance 2 from a vertex of X which is annihilated by v. Hence, for every graph G, we have

Gathering these inequalities with these ones obtained in Section 3 and relating the perfect neighborhood sets to the 2-packings and the 2-coverings, we get for all G:

$$\gamma_2(G) \leq \left\{ \begin{array}{l} ra(G) \leq \left\{ \begin{array}{l} rai(G) \\ er(G) \end{array} \right\} \leq ir(G) \\ \theta(G) \leq \theta_i(G) \leq \rho L(G) \leq \rho(G) \end{array} \right\} \leq \gamma(G).$$

In general the parameters of the two intermediary lines cannot be ranged in a linear order (we saw in Section 2 that $\theta(G)$ can be smaller or greater than $ir(G)$ and we only know that $\rho(G)$ $3ra(G)/2$ (Cockayne *et al.* 1998 a), However, if G is a tree then $(G) =$ (G) by Meir and Moon (1975) and it is proved in Cockayne *et al.* (1998 b) that $\rho L(G)$ $ra(G)$. Therefore, every tree T satisfies:

$$\gamma_2(T) \leq \theta(T) \leq \theta_i(T) \leq \rho_L(T) \leq ra(T) \leq \quad \leq ir(T)$$
$$\leq \rho(T) = \gamma(T).$$

To conclude, it has been observed in Cockayne *et al.* (1997) and in Cockayne *et al.* (1998 b) that several known lower bounds on $ir(G)$ still hold, with exactly the same proof, for $er(G)$ or for $ra(G)$. Therefore, in some circumstances and for some problems, the concept of irredundance be-comes redundant and can be replaced by a concept of annihilation.

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